

AMALGAMATION OF RINGS DEFINED BY BEZOUT-LIKE CONDITIONS

MOHAMMED KABBOUR AND NAJIB MAHDOU

ABSTRACT. Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B . In this paper, we investigate the transfer of notions elementary divisor ring, Hermite ring and Bézout ring to the amalgamation $A \bowtie^f J$. We provide necessary and sufficient conditions for $A \bowtie^f J$ to be an elementary divisor ring where A and B are integral domains. In this case it is shown that $A \bowtie^f J$ is an Hermite ring if and only if it is a Bézout ring. In particular, we study the transfer of the previous notions to the amalgamated duplication of a ring A along an A -submodule E of $Q(A)$ such that $E^2 \subseteq E$.

1. INTRODUCTION

All rings considered in this paper are assumed to be commutative, and have identity element and all modules are unitary.

A ring R is called an elementary divisor ring (resp. Hermite ring) if for every matrix M over R there exist non singular matrices P, Q such that PMQ (resp. MQ) is a diagonal matrix (resp. triangular matrix). It proved in [7] that a ring R is an Hermite ring if and only if for all $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1d$, $b = b_1d$, and $Ra_1 + Rb_1 = R$. A ring is a Bézout ring if every finitely generated ideal is principal. It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is a Bézout ring. Following Kaplansky [10] a ring R is said to be a valuation ring if for any two elements in R , one divides the other. Kaplansky proved that any valuation ring is an elementary divisor ring.

Let A and B be rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In [4] the amalgamation of A with B along J with respect to f is the sub-ring of $A \times B$ defined by:

$$A \bowtie^f J = \{(a, f(a) + j) ; a \in A, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [5], [2] and in [6]. Moreover, several classical construction such as $A + xK[x]$ and $A + xK[[x]]$ can be

2000 *Mathematics Subject Classification.* 13D05, 13D02.

Key words and phrases. Elementary divisor ring, Hermite ring, Bézout ring and amalgamation of rings.

studied as particular case of the amalgamation.

We denote $Q(A)$ the total ring of quotients of A . Let E be an A -submodule of $Q(A)$ such that $E^2 \subseteq E$, $A + E$ is a sub-ring of $Q(A)$ and E is an ideal of $A + E$. The amalgamated duplication of A along E :

$$A \bowtie E = \{(a, a + e) ; a \in A, e \in E\}$$

is also a particular case of the amalgamation of A with $A + E$ along E with to respect f , where $f : A \hookrightarrow A + E$ is the inclusion map. In fact, the amalgamated duplication of A along E can be studied in the frame of amalgamation construction. Our aim in this paper is to give a characterization for $A \bowtie^f J$ to be an elementary divisor ring, an Hermite ring and a Bézout ring.

2. MAIN RESULTS

The set of all $n \times n$ matrices with entries from a ring R will be denoted by $\mathcal{M}_n(R)$. We will let $\mathcal{GL}_n(R)$ denote the units in $\mathcal{M}_n(R)$. Let A and B be rings, for every matrix $M = ((a_{i,j}, b_{i,j}))_{1 \leq i,j \leq n} \in \mathcal{M}_n(A \times B)$ we shall use the notation $M_a = (a_{i,j})_{1 \leq i,j \leq n}$, $M_b = (b_{i,j})_{1 \leq i,j \leq n}$ and $M = M_a \times M_b$. Let $M, N \in \mathcal{M}_n(A \times B)$, it is easy to see that the product MN of M and N is giving by $MN = (M_a N_a) \times (M_b N_b)$.

The following lemma will be useful to provide us many statements in this paper.

Lemma 2.1. *Let A and B a pair of integral domains, $f : A \longrightarrow B$ a ring homomorphism and let J be a proper ideal of B .*

- (1) *If $A \bowtie^f J$ is a Bézout ring then $f(A) \cap J = 0$.*
- (2) *If $A \bowtie^f J$ is a Bézout ring and f is not injective then $J = 0$.*

Proof. (1) Suppose the statement is false i.e $f(A) \cap J \neq 0$, and choose an element $a \in A$ such that $0 \neq f(a) \in J$. Then $(0, f(a))$ is an element of $A \bowtie^f J$. Since $A \bowtie^f J$ is a Bézout ring the ideal generated by $(0, f(a))$ and $(a, f(a))$ is principal. Hence, there exists $(d, f(d) + j) \in A \bowtie^f J$ such that

$$(a, f(a)) (A \bowtie^f J) + (0, f(a)) (A \bowtie^f J) = (d, f(d) + j) (A \bowtie^f J).$$

So, there exist $(b, f(b) + x), (c, f(c) + y), (\alpha, f(\alpha) + s)$ and $(\beta, f(\beta) + t)$ in $A \bowtie^f J$ such that

$$\begin{cases} (0, f(a)) = (d, f(d) + j)(b, f(b) + x) \\ (a, f(a)) = (d, f(d) + j)(c, f(c) + y) \\ (d, f(d) + j) = (0, f(a))(\alpha, f(\alpha) + s) + (a, f(a))(\beta, f(\beta) + t). \end{cases}$$

It follows that $d \neq 0$ since $a = cd$ and $f(a) \neq 0$. Also $b = 0$ since $bd = 0$ and A is an integral domain. From the previous equalities we deduce that $f(a) = (f(d)+j)x = (f(d)+j)(f(c)+y)$ and $f(d)+j = f(a)(f(\alpha)+f(\beta)+s+t)$.

Multiplying the above equality by x , we get that $1 = x(f(\alpha) + f(\beta) + s + t)$ since B is an integral domain. We conclude that x is a unit, but $x \in J$ hence $J = B$ which is absurd. We have the desired result.

(2) Assume that $J \neq 0$ and let $0 \neq u \in J$. Since f is non injective there exists $0 \neq a \in \ker f$. From the assumption we can write

$$(a, u) \left(A \bowtie^f J \right) + (0, u) \left(A \bowtie^f J \right) = (d, f(d) + j) \left(A \bowtie^f J \right)$$

for some $(d, f(d) + j) \in A \bowtie^f J$. With similar proof as in the statement (1), we get that $J = B$. This completes the proof of Lemma 2.1. \square

Lemma 2.2. *The following assertions holds:*

- (1) *Let A and B be rings. Then $A \times B$ is an elementary divisor ring if and only if so A and B .*
- (2) *Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . If $A \bowtie^f J$ is an elementary divisor ring then so is A and $f(A) + J$.*

Proof. (1) We begin by showing that if $M \in \mathcal{M}_n(A \times B)$ then M is invertible if and only if so is M_a and M_b . We put $M = ((a_{i,j}, b_{i,j}))_{1 \leq i,j \leq n}$. The determinant of M is giving by

$$\det M = \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \prod_{i=1}^n (a_{i,\sigma(i)}, b_{i,\sigma(i)})$$

where \mathcal{S}_n denotes the set of all permutations on n letters and $\varepsilon(\sigma)$ denotes the sign of σ , for every $\sigma \in \mathcal{S}_n$. Thus $\det M = (\det M_a, \det M_b)$. We say that M is invertible if and only if $\det M$ is a unit. Then we have the desired result. Assume that $A \times B$ is an elementary divisor ring. Let $U \in \mathcal{M}_n(A)$ then $U \times 0$ is equivalent to a diagonal matrix D with entries from $A \times B$. There is some $P, Q \in \mathcal{GL}_n(A \times B)$ such that $P(U \times 0)Q = D$. It follows that $P_a U Q_a = D_a$ and so A is an elementary divisor ring. By the same way we get that B is an elementary divisor ring.

Conversely, assume that A and B are elementary divisor rings and let $M \in \mathcal{M}_n(A \times B)$. Then there exist two invertible matrices P_1 and Q_1 (resp., P_2 and Q_2) and a diagonal matrix D (resp., Δ) with entries from A (resp., B) such that $P_1 M_a Q_1 = D$ (resp., $P_2 M_b Q_2 = \Delta$). It follows that

$$(P_1 \times P_2)M(Q_1 \times Q_2) = (P_1 M_a Q_1) \times (P_2 M_b Q_2) = D \times \Delta,$$

which is a diagonal matrix. From the previous part of the proof $P_1 \times P_2, Q_1 \times Q_2 \in \mathcal{GL}_n(A \times B)$. This completes the proof of (1).

(2) Let $U = (a_{i,j})_{1 \leq i,j \leq n} \in \mathcal{M}_n(A)$ and let M be the matrix defined by $M = ((a_{i,j}, f(a_{i,j})))_{1 \leq i,j \leq n}$ with entries from $A \bowtie^f J$. We have the equality

$U = M_a$. Since $A \bowtie^f J$ is an elementary divisor ring M is equivalent to a diagonal matrix. From the previous part of the proof we deduce that there exist P and Q in $\mathcal{GL}_n(A)$ such that PUQ is a diagonal matrix. Therefore A is an elementary divisor ring. With similar proof as in above we get that $f(A) + J$ is an elementary divisor ring. \square

Remark 2.3. Let $f : A \longrightarrow B$ be a ring homomorphism, J an ideal of B and let $M \in \mathcal{M}_n(A \bowtie^f J)$. Then M is invertible if and only if so is M_a and M_b .

Proof. It is sufficient to prove that if $M_a \in \mathcal{GL}_n(A)$ and $M_b \in \mathcal{GL}_n(B)$ then $M_a^{-1} \times M_b^{-1} \in \mathcal{GL}_n(A \bowtie^f J)$. Let $(a, f(a) + j) \in A \bowtie^f J$ which is a unit in the ring $A \times B$. We put $x = -f(a^{-1})(f(a) + j)^{-1}j$. Since J is an ideal of B , $x \in J$. It is easy to get the following equality

$$(a^{-1}, f(a^{-1}) + x)(a, f(a) + j) = (1, 1).$$

Thus $(a, f(a) + j)^{-1} \in A \bowtie^f J$. We say that $\det M$ is an element of $A \bowtie^f J$ which is a unit in $A \times B$, therefore $(\det M)^{-1} \in A \bowtie^f J$. Consequently, $M^{-1} \in \mathcal{M}_n(A \bowtie^f J)$. \square

Theorem 2.4. Let A and B a pair of integral domains, $f : A \longrightarrow B$ a ring homomorphism and let J be an ideal of B .

(1) Assume that f is injective.

- If $J = B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so is A and B .
- If $J \neq B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so is $f(A) + J$ and $f(A) \cap J = 0$.

(2) Assume that f is not injective. Then $A \bowtie^f J$ is an elementary divisor ring if and only if one of the following conditions holds:

- $J = 0$ and A is an elementary divisor ring.
- $J = B$ and (A, B) is a pair of elementary divisor rings.

Proof. (1) Two cases will be considered.

case 1: If $J = B$ then $A \bowtie^f J = A \times B$. By applying condition (1) of Lemma 2.2, we get that $A \bowtie^f J$ is an elementary divisor ring if and only if so is A and B .

case 2: If $J \neq B$ and $A \bowtie^f J$ is elementary divisor ring then $f(A) \cap J = 0$ by Lemma 2.1 since every elementary divisor ring is a Bézout ring. On the other hand $f(A) + J$ is an elementary divisor ring by Lemma 2.2

Conversely, assume that $f(A) + J$ is an elementary divisor ring and $f(A) \cap J = 0$. We claim that the natural projection $p_B : A \bowtie^f J \longrightarrow f(A) + J$ ($p_B(a, f(a) + j) = f(a) + j$) is a ring isomorphism. Indeed,

$$f(a) + j = 0 \Rightarrow f(a) = j = 0 \Rightarrow a = 0$$

The conclusion is now straightforward.

(2) Assume that $A \bowtie^f J$ is an elementary divisor ring. By using Lemma 2.1, we get that $J = 0$ or $J = B$. In the first case $A \bowtie^f J \simeq A$, then A is an elementary divisor ring. In the second case $A \bowtie^f J = A \times B$. Hence A and B are elementary divisor rings by Lemma 2.2. The converse of (2) is an immediate consequence of Lemma 2.2. \square

Theorem 2.4 enriches the literature with a new example of a non valuation elementary divisor ring.

Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . It is easy to see that: if $A \bowtie^f J$ is a valuation ring then so is A .

Example 2.5. Let A be an elementary divisor domain which is not a valuation ring (for instance $A = \mathbb{Z}$), and let K its field of fractions. Let $K[[x]]$ denote the ring of formal power series over K in an indeterminate x . By [[9], Example1 p.161], $A + (xK[[x]])$ is an elementary divisor ring. We conclude that $A \bowtie^i (xK[[x]])$, where i is the inclusion map of A into $K[[x]]$, is an elementary divisor ring. On the other hand $A \bowtie^i (xK[[x]])$ is not a valuation ring. Thus $\mathbb{Z} \bowtie^i (x\mathbb{Q}[[x]])$ is an elementary divisor ring which is not a valuation ring.

Corollary 2.6. *Let A be an integral domain, K its quotient field and let E be a nonzero A -submodule of K such that $E^2 \subseteq E$. Then $A \bowtie E$ is an elementary divisor ring if and only if so is A and $A \subseteq E$.*

Proof. We first prove that: Any ring R' between an elementary divisor ring R and its total ring $Q(R)$, is also an elementary divisor ring.

Let $M = \left(\frac{a_{i,j}}{d} \right)_{1 \leq i,j \leq n} \in \mathcal{M}_n(R')$, where $a_{i,j} \in R$ for each $1 \leq i, j \leq n$ and d is a nonzero divisor element of R . There is some invertible matrices P and Q with entries from R such that $P(a_{i,j})_{1 \leq i,j \leq n}Q$ is a diagonal matrix. Set $P(a_{i,j})_{1 \leq i,j \leq n}Q = \text{diag}(\lambda_1, \dots, \lambda_n)$. Multiplying this equality by $\frac{1}{d}$, we get that $PMQ = \text{diag}\left(\frac{\lambda_1}{d}, \dots, \frac{\lambda_n}{d}\right)$. Since $PMQ \in \mathcal{M}_n(R')$ the result follows.

Now suppose that A is an elementary divisor ring and $A \subseteq E$. We have $A \bowtie E = A \times E$. From the previous part of the proof and condition (1) of Lemma 2.2, we get that $A \bowtie E$ is an elementary divisor ring. Conversely, assume that $A \bowtie E$ is an elementary divisor ring. We have $A \bowtie E = A \bowtie^i E$, where $i : A \hookrightarrow A + E$ is the inclusion map. By using the condition (1) of Theorem 2.4, we obtain the following result:

- If $E = A + E$ (i.e $A \subseteq E$) then A and $A + E$ are elementary divisor rings.
- Otherwise $(A + E) \cap E = 0$ and $A + E$ is elementary divisor ring.

From the assumption $(A + E) \cap E \neq 0$ since $E \subseteq A + E$. We conclude that $A \subseteq E$ and A is an elementary divisor ring. \square

Example 2.7. Let A be an integral domain and let I be a nonzero ideal of A . Then $A \bowtie I$ is an elementary divisor ring if and only if so is A and $I = A$.

Lemma 2.8. *Let A and B be a pair of rings. Then:*

- (1) *$A \times B$ is a Bézout ring if and only if so is A and B .*
- (2) *$A \times B$ is an Hermite ring if and only if so is A and B .*

Proof. (1) Suppose that A and B are Bézout rings and let I be a finitely generated ideal of $A \times B$. There is some ideal I_1 of A and I_2 of B such that $I = I_1 \times I_2$. If the subset $\{(a_1, b_1), \dots, (a_n, b_n)\}$ of $A \times B$ generate I then $I_1 = Aa_1 + \dots + Aa_n$. Thus I_1 is a principal ideal of A . There exists $a \in I_1$ such that $I_1 = Aa$. By the same way, we get that there exists $b \in I_2$ such that $I_2 = Bb$. We deduce that $I = (A \times B)(a, b)$. Conversely assume that $A \times B$ is a Bézout ring. Let J_1 be a finitely generated ideal of A and let $J = J_1 \times 0$. Then J is also finitely generated ideal of $A \times B$, we get that J is a principal ideal of $A \times B$. Hence so is J_1 , therefore A is a Bézout ring. Also B is a Bézout ring since $A \times B \simeq B \times A$.

(2) Assume that $A \times B$ is an Hermite ring. Let $a, a' \in A$ then there exist $(a_1, b_1), (a'_1, b'_1), (d, \delta) \in A \times B$ such that

$$\begin{cases} (a, 0) = (a_1, b_1)(d, \delta) \\ (a', 0) = (a'_1, b'_1)(d, \delta) \\ A \times B = (a_1, b_1)(A \times B) + (a'_1, b'_1)(A \times B). \end{cases}$$

Let $(\alpha, \beta), (\alpha', \beta') \in A \times B$ such that $(\alpha, \beta)(a_1, b_1) + (\alpha', \beta')(a'_1, b'_1) = (1, 1)$. It follows that $a = a_1d$, $a' = a'_1d$ and $\alpha a_1 + \beta a'_1 = 1$. We conclude that A and B is a pair of Hermite rings since $A \times B \simeq B \times A$. The converse of the statement is obvious. \square

Theorem 2.9. *Let A and B be a pair of integral domains, J an ideal of B and let $f : A \rightarrow B$ be an injective ring homomorphism. Then the following properties are equivalent:*

- (1) *$A \bowtie^f J$ is an Hermite ring.*
- (2) *$A \bowtie^f J$ is a Bézout ring.*
- (3) *One of the following conditions holds:*
 - *$J = B$, A and B are Bézout rings.*
 - *$J \neq B$, $f(A) \cap J = 0$ and $f(A) + J$ is a Bézout ring.*

Proof. (1) \Rightarrow (2): Clear.

(2) \Rightarrow (3): Assume that $J \neq B$. By Lemma 2.1 $f(A) \cap J = 0$. Then the natural projection $p_B : A \bowtie^f J \rightarrow f(A) + J$; $(p_B(a, f(a) + j) = f(a) + j)$ is

a ring isomorphism since f is injective. Therefore $f(A) + J$ is a Bézout ring. If $J = B$ then A and B are Bézout rings by the condition (1) of Lemma 2.8 since $A \bowtie^f J = A \times B$.

(3) \Rightarrow (1): If $J = B$ then A and B are Hermite rings since every Bézout domain is an Hermite ring. Hence $A \bowtie^f J = A \times B$ is an Hermite ring. Now we assume that $J \neq B$. Then $A \bowtie^f J \simeq f(A) + J$ and so $A \bowtie^f J$ is a Bézout domain. This completes the proof of Theorem 2.9. \square

Example 2.10. Let A be a Bézout domain, K its quotient field, and let $K[[x]]$ denote the ring of formal power series over K in an indeterminate x . Then $A \bowtie^i (xK[[x]])$, where $i : A \hookrightarrow K[[x]]$ is the inclusion map, is an Hermite ring.

Proof. Let $f = \sum_{n=0}^{\infty} a_n x^n$, $g = \sum_{n=0}^{\infty} b_n x^n$ be nonzero elements of $R = A + (xK[[x]])$, and let p (resp. q) denote the least integer such that $a_p \neq 0$ (resp., $b_q \neq 0$). We can write $f = a_p x^p (1 + x f_1)$ and $g = b_q x^q (1 + x g_1)$, where $f_1, g_1 \in K[[x]]$. Since $1 + x f_1$, $1 + x g_1$ are units of R ,

$$fR + gR = a_p x^p R + b_q x^q R.$$

If $p < q$ (resp., $q < p$) then $fR + gR = a_p x^p R$ (resp., $b_q x^q R$). Suppose that $p = q$ and write $a_p = \frac{a}{d}$ and $b_q = \frac{b}{d}$ for some nonzero elements a, b, d of A (where $d = 1$ if $p = q = 0$). By the assumption there exist $c, a', b' \in A$ such that $a = a'c$, $b = b'c$ and $a'A + b'A = A$. It is easy to get that $fR + gR = \frac{c}{d} x^p R$. This completes the proof that $A \bowtie^i (xK[[x]])$ is an Hermite ring. \square

Corollary 2.11. Let A be an integral domain, K its quotient field and let E be a nonzero A -submodule of K such that $E^2 \subseteq E$. Then the following statements are equivalent:

- (1) $A \bowtie E$ is an Hermite ring.
- (2) $A \bowtie E$ is a Bézout ring.
- (3) A is a Bézout ring and $A \subseteq E$.

Proof. (2) \Rightarrow (3): Let $0 \neq \frac{a}{b} \in E$. Then $0 \neq a \in A \cap E$ and so $A \cap E \neq 0$. By applying Theorem 2.9, we get that $A + E = E$ and $(A, A + E)$ is a pair of Bézout rings. It follows that $A \subseteq E$ and A is a Bézout ring.

(3) \Rightarrow (1): By applying Lemma 2.8 and the condition (3) of Theorem 2.9 it is sufficient to prove that every ring between a Bézout domain and its quotient field is also Bézout domain. Let R be a Bézout domain and let R'

be a ring such that $R \subseteq R' \subseteq qf(R)$. Let $\frac{a}{d}, \frac{b}{d} \in R'$ then we can write

$$\begin{cases} a = a'c \\ b = b'c \\ \alpha a' + \beta b' = 1 \end{cases}$$

for some elements a', b', c, α, β in R . Hence $\frac{c}{d} = \alpha \frac{a}{d} + \beta \frac{b}{d}$ is an element of R' . Thus $\frac{c}{d} \in R'$ and $R' \frac{a}{d} + R' \frac{b}{d} \subseteq R' \frac{c}{d}$. On the other hand, we have:

$$\frac{c}{d} \in R' \frac{a}{d} + R' \frac{b}{d} \subseteq R' \frac{a}{d} + R' \frac{b}{d}.$$

It follows that $R' \frac{a}{d} + R' \frac{b}{d} = R' \frac{c}{d}$. Finally, R' is a Bézout domain. \square

Example 2.12. Let A be an integral domain and let I be a nonzero ideal of A . Then $A \bowtie I$ is a Bézout ring if and only if so is A and $I = A$.

Theorem 2.13. Let A and B be a pair of integral domains, J an ideal of B and let $f : A \rightarrow B$ be a non injective ring homomorphism. Then the following statements are equivalent:

- (1) $A \bowtie^f J$ is an Hermite ring.
- (2) $A \bowtie^f J$ is a Bézout ring.
- (3) One of the following conditions holds:
 - $J = B$, A and B are Bézout rings.
 - $J = 0$, and A is a Bézout ring.

Proof. (2) \Rightarrow (3): By applying condition (2) of Lemma 2.1, we get that $J = 0$ or $J = B$. If $J = 0$ then $A \bowtie^f J \simeq A$ otherwise $A \bowtie^f J = A \times B$. By using Lemma 2.8, we have the desired implication.

(3) \Rightarrow (1): If $J = 0$ then $A \bowtie^f J \simeq A$ and A is an Hermite ring (since A is an integral domain). If $J = B$ then $A \bowtie^f J = A \times B$ is an Hermite ring by condition (2) of Lemma 2.8. \square

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MOHAMMED KABBOUR, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202, UNIVERSITY S.M. BEN ABDELLAH FEZ, MOROCCO.

E – mail address : mkabbour@gmail.com

NAJIB MAHDOU, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202, UNIVERSITY S.M. BEN ABDELLAH FEZ, MOROCCO.

E – mail address : mahdou@hotmail.com